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Basically Full Ideals in Local Rings

William J. Heinzer

Department of Mathematics, Purdue University, West Lafayette, Indiana 47907

and

Louis J. Ratliff Jr. and David E. Rush

Department of Mathematics, University of California, Riverside, California 92521

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Let A be a finitely generated module over a (Noetherian) local ring (R, M). We say that a nonzero submodule B of A is *basically full* in A if no minimal basis for B can be extended to a minimal basis of any submodule of A properly containing B. We prove that a basically full submodule of A is M-primary, and that the following properties of a nonzero M-primary submodule B of A are equivalent: (a) B is basically full in A; (b) $B = (MB) :_A M$; (c) MB is the irredundant intersection of $\mu(B)$ irreducible ideals; (d) $\mu(C) \leq \mu(B)$ for each cover C of B. Moreover, if B is an M-primary submodule of A, then $B^* := (MB) :_A M$ is the smallest basically full submodule of A containing B and $B \mapsto B^*$ is a semiprime operation on the set of nonzero M-primary submodules B of A. We prove that all nonzero M-primary ideals are closed with respect to this operation if and only if M is principal. In relation to the closure operation $B \mapsto B^*$, we define and study the bf-reductions of an M-primary submodule D of A; that is, the M-primary submodules C of D such that $C \subseteq D \subseteq C^*$. If $G(M)$ denotes the form ring of R with respect to M and $G^+(M)$ its maximal homogeneous ideal, we prove that $M^n = (M^n)^*$ for all (resp. for all large) positive integers n if and only if $\text{grade}(G^+(M)) > 0$ (resp. $\text{grade}(M) > 0$). For a regular local ring (R, M), we consider the M-primary monomial ideals with respect to a fixed regular system of parameters and determine necessary and sufficient conditions for such an ideal to be basically full.

Key Words: basically full ideal; basis of an ideal; closure operation; cover of an ideal; form ring; injective envelope; integral closure of an ideal; irreducible submodule; monomial ideal; Noetherian ring; R-sequence; reduction of an ideal; regular local ring; semiprime operation; socle; superficial element.

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1. INTRODUCTION

The investigation of the number of generators of ideals and modules over a local ring has a rich history; [1, 3, 11, 24, 25, 29, 30, 31] are just a few of the many references on this topic. We consider here the ideals of a local ring and the submodules of a finitely generated module over a local ring which satisfy a maximal property with respect to extension of a minimal basis.

Let A be a finitely generated module over a local ring (R, M). For a nonzero submodule B of A, Nakayama's lemma implies that elements $b_1, \ldots, b_n \in B$ are a minimal basis for B if and only if their images in $B/(MB)$ are a basis for the vector space $B/(MB)$ over the field R/M; see, for example [14, (4.1), 12, p. 8, or 11, p. 104]. We say that B is basically full in A if no minimal basis of B can be extended to a minimal basis of any submodule of A that properly contains B.

In Section 2 we show in Theorem 2.6 that the basically full submodules of A are M-primary. We give in Theorems 2.4, 2.12, and 2.17 several characterizations of the basically full submodules of A. In particular, for an M-primary submodule B of A, we show that B is basically full if and only if $B = (MB) :_A M$ if and only if there are $\mu(B)$ submodules in an irredundant representation of MB as an intersection of irreducible submodules of A if and only if $\mu(C) \leq \mu(B)$ for all covers C of B in A.

We show in Theorem 3.3 that one basically full ideal in a local ring yields other basically full ideals in related local rings. In Theorem 4.2 we prove that the function $B \mapsto B^* = (MB) :_A M$ is a closure operation on the set of nonzero M-primary submodules of A. Thus B^*, the basically full closure of B, is the smallest basically full submodule of A containing B.

In Section 5 we introduce the concept of a bf-reduction of a nonzero M-primary submodule of A. If $C \subseteq B$ are nonzero M-primary submodules of A, we call C a bf-reduction of B if $B \subseteq C^*$, and we say that B is bf-basic if it has no proper bf-reductions. We show that most of the standard properties of ordinary reductions also hold for bf-reductions and that B^* is the largest submodule C of A with the following property: if D, E are R-submodules of A with $B \subseteq D \subseteq E \subseteq C$, then each minimal basis of D extends to a minimal basis of E.

In Section 6 we consider implications of the fact that the basically full closure B^* of an open irreducible submodule B is either B or the unique cover $B :_A M$ of B.

In Theorem 7.1, we prove that M^n is basically full for every positive integer n if and only if $\text{Grade}(G^+(M)) > 0$, where $G^+(M)$ is the maximal homogeneous ideal of the form ring $G(M)$. We prove in Theorem 7.2 that M^n is basically full for all large integers n if and only if $\text{Grade}(M) > 0$. In Theorem 7.4, we show that if altitude$(R) > 0$, then $M^n I + ((0) :_R M^k)$

is basically full for all open ideals I and for all large integers n and k. In Theorem 7.5, we prove that every nonzero M-primary ideal of (R, M) is basically full if and only if R is a principal ideal ring.

In Section 8 we consider M-primary ideals in a regular local ring (R, M), which are generated by monomials in a fixed regular system of parameters of R. For such an ideal I, we observe in Proposition 8.2 that the basically full closure I^* of I is again a monomial ideal. We prove in Corollary 8.4 that to determine whether I is basically full it suffices to show that a minimal monomial basis of I does not extend to a minimal monomial basis of any properly larger monomial ideal of R.

Finally, in Section 9, several examples are given which illustrate some of the results of this paper. These examples are all in a regular local ring of altitude two and concern monomial ideals in a fixed regular system of parameters.

Throughout the paper we use $\subset$ to denote *proper* containment, and we use $\mu(A)$ to denote the number of elements in a minimal basis of a finitely generated module A over a local ring. A general reference for our notation and terminology is [12].

2. BASIC PROPERTIES OF BASICALLY FULL SUBMODULES

Let (R, M) be a (Noetherian) local ring and A a finitely generated R-module. In this section we define basically full submodules of A, show that such modules are M-primary, and give several characterizations of basically full submodules of A. In particular, a submodule B of A is basically full if and only if MB is an irredundant intersection of $\mu(B)$ irreducible submodules of A (which is the smallest possible number) if and only if $\mu(C) \leq \mu(B)$ for each cover C of B. It is also shown that if $\mathrm{ht}(M) > 0$, then each integrally closed M-primary ideal is basically full.

DEFINITION 2.1. A nonzero submodule B of a finitely generated R-module A is said to be *basically full* in A if no minimal basis of B can be extended to a minimal basis of a submodule of A that properly contains B. For an ideal $I \subseteq M$ of R, we say I is basically full if I is basically full in R.

Concerning the terminology "basically full," when we first began looking at this concept we used "basically maximal." However, we later changed to "basically full" because of Rees' terminology "M-full" for the closely related concept defined in (2.2.1).

Remark 2.2. (2.2.1) Let (R, M) be a local ring. The concept of an M-full ideal of R is originally due to D. Rees (unpublished). It appears in work of Goto [5] and Watanabe [29–31]. The definition is as follows

[29, p. 102]: (1) If the residue field R/M is infinite, an ideal $I \subseteq M$ is *M-full* if there exists $y \in M$ such that $(MI) :_R yR = I$. (2) If R/M is not necessarily infinite, let R' be a local ring which is faithfully flat over R with infinite residue field and with MR' as the maximal ideal. Then an ideal I of R is called M-full if IR' is M-full in the sense of (1). Let $M = (m_1, \ldots, m_n)R$, let $R(X_1, \ldots, X_n) = R[X_1, \ldots, X_n]_{MR[X_1,\ldots,X_n]} := R'$, and let $Y = m_1X_1 + \cdots + m_nX_n$. It follows [29] that, in general, an ideal $I \subseteq M$ of R is M-full if and only if $(MIR') :_{R'} YR' = IR'$. An ideal I of R is said to have the *Rees property* if $\mu(J) \leq \mu(I)$ for each ideal J of R that contains I. It is clear that an ideal which has the Rees property is basically full, and it is shown in [29, Theorem 3] that an M-primary M-full ideal has the Rees property. Thus the M-primary ideals that are M-full are a source of examples of basically full ideals. There are, however, basically full ideals which are not M-full (see Example 9.1 below).

(2.2.2) It is also shown in [29, Theorem 4] that if (R, M) is a regular local ring of altitude two, then M^n is M-full for each positive integer n. We consider the structure of these ideals in Examples 9.2 and 9.3.

(2.2.3) In Definition 2.1, if A is nonzero, we view A to be basically full in A; in particular, R is a basically full ideal of itself. Also, if $A = (0)$, then we view A to be basically full in A, but if $A \neq (0)$, then (0_A) is not basically full in A (since the empty set (it is minimal basis) can be extended to a minimal basis of every nonzero submodule of A).

In Theorem 2.3 we observe that if one minimal basis of a submodule B is extendable, then all minimal bases of B are extendable.

THEOREM 2.3. *Let A be a Noetherian module over a local ring (R, M) and $B \subset C$ be nonzero submodules of A. Then the following are equivalent*:

(2.3.1) *Some minimal basis for B is part of a minimal basis for C.*

(2.3.2) *Every minimal basis for B is part of a minimal basis for C.*

(2.3.3) *$MB = (MC) \cap B$.*

Proof. (2.3.3) $\Rightarrow$ (2.3.2) If $MB = (MC) \cap B$, then $B/(MB) = B/((MC) \cap B) \subset C/(MC)$, and each basis for the R/M vector space $B/(MB)$ can be extended to a basis of the R/M vector space $C/(MC)$. It follows that each minimal basis of B can be extended to a minimal basis of C.

(2.3.2) $\Rightarrow$ (2.3.1) Clear.

(2.3.1) $\Rightarrow$ (2.3.3) Assume some minimal basis $b_1, \ldots, b_g$ of B extends to a minimal basis $b_1, \ldots, b_g, b_{g+1}, \ldots, b_k$ of C. Let $x \in (MC) \cap B$. Then $x = \sum_{i=1}^{g} r_i b_i$ for some $r_1, \ldots, r_g$ in R. Since $\sum_{i=1}^{g} r_i b_i \in MC$ and $b_1 + MC, \ldots, b_g + MC$ are linearly independent over R/M, $r_i \in M$ for each i. Thus $x \in MB$. ∎

Theorem 2.3 yields the following characterization of the basically full submodules of a Noetherian module.

THEOREM 2.4. *Let A be a Noetherian module over a local ring (R, M) and let B be a nonzero submodule of A. Then B is basically full in A if and only if $MB \subset (MC) \cap B$ for each submodule C of A that properly contains B.*

Proof. This follows from Theorem 2.3. ∎

We also record the following corollary to Theorem 2.4.

COROLLARY 2.5. *Let A be a Noetherian module over a local ring (R, M). A nonzero submodule B of A is basically full in A if and only if the image of B in $A/(MB)$ is basically full in $A/(MB)$.*

We show in Theorem 2.6 that basically full submodules are M-primary. In contrast, M-full ideals need not be M-primary, since any prime ideal is M-full, by [29, Remark 1].

THEOREM 2.6. *If A is a Noetherian module over a local ring (R, M) and if $B \neq A$ is a basically full submodule in A, then B is an M-primary submodule of A.*

Proof. Assume that B is not M-primary. By Artin–Rees [11, p. 151 or 12, p. 59], for all sufficiently large positive integers n we have $(M^n A) \cap B \subseteq MB$. For n with this property, let $C = B + M^n A$. Then $C \neq B$ since B is not M-primary, and $(MC) \cap B = (M(B + M^n A)) \cap B = (MB + M^{n+1} A) \cap B = MB + ((M^{n+1} A) \cap B)$ (by modularity) $= MB$. Hence by Theorem 2.4, B is not basically full in A. ∎

COROLLARY 2.7. *Let I be a non-open ideal in a local ring (R, M) and let J be an ideal of R which is not contained in* $\operatorname{Rad}(I)$. *For each positive integer n let $K_n = I + M^n J$. Then the following hold for all large integers n:*

(2.7.1) *Each minimal basis of I can be extended to a minimal basis of K_n.*

(2.7.2) $(MK_n) \cap I = MI$.

Proof. This follows immediately from Theorem 2.6 and its proof. ∎

Remark 2.8. Nakayama's lemma also applies to finitely generated modules over a non-Noetherian quasilocal ring (R, M). As in the Noetherian case, it is natural to define a nonzero finitely generated ideal $I \subseteq M$ to be basically full if no minimal basis of I can be extended to a minimal basis of a finitely generated ideal of R that properly contains I. For a non-Noetherian quasilocal ring (R, M), it can happen that an ideal $I \subseteq M$ is basically full, but is not M-primary. For example, if R is a valuation domain, then every nonzero principal ideal $I \subseteq M$ is basically full. Thus if altitude$(R) > 1$, then there exist basically full ideals of R which are not M-primary.

We develop additional applications of Theorem 2.6 in the following context: for a submodule B of the R-module A, we consider submodules of the form $(MB) :_A M = \{a \in A \mid Ma \subseteq MB\}$. In this regard, if I is an ideal of R and ht$(M) > 0$, then $I \subseteq (MI) :_R M \subseteq \cup\{(M^n I) :_R M^n \mid n$ is a positive integer$\} \subseteq I_a$ by [19, (3.2)], where I_a denotes the integral closure of I in R.

We use the following definitions and remark.

DEFINITION 2.9. (2.9.1) The *socle* of a Noetherian module A over a local ring (R, M) is defined to be $(0_A) :_A M$. We denote by sdim(A) the dimension of the socle of the R-module A as a vector space over R/M.

(2.9.2) If $B \subseteq C$ are submodules of a Noetherian module A over a local ring (R, M), then C is a *cover* of B if $C/B \cong R/M$.

(2.9.3) A submodule B of an R-module A is said to be *essential* in A if $C \cap B \neq (0)$ for each nonzero submodule C of A.

(2.9.4) A module E containing an R-module A is said to be an *injective envelope* of A if A is essential in E and E is injective.

(2.9.5) A submodule B of an R-module A is said to be *irreducible* if $C \cap D \neq B$ for each pair of submodules C, D of A which properly contain B.

Remark 2.10. If B is a submodule of a Noetherian module A over a local ring (R, M) and if $C = (MB) :_A M$, then $MC = M((MB) :_A M) = MB$, and hence $(MC) :_A M = (MB) :_A M = C$. Also, if $D = B :_A M$, then $(MD) :_A M = (M(B :_A M)) :_A M \subseteq B :_A M = D \subseteq (MD) :_A M$, so $D = (MD) :_A M$.

We believe the following lemma to be known; see [26, Theorem 4.9, p. 91].

LEMMA 2.11. *Let C be a module of finite length over the local ring (R, M). Let $C_1 \cap \cdots \cap C_j = (0_C)$, where each C_i is irreducible and the intersection is irredundant, and let $E(C)$ be an injective envelope of C. Then $E(C) \cong E(R/M)^t$, where $j = t =$ sdim(C).*

Proof. Let $F = R/M$ and let $s =$ sdim(C). Then $E(F)^s \cong E(F^s) \cong E((0) :_C M) = E(C)$ (since the inclusion $(0) :_C M \subseteq C$ is essential) $= \oplus_{i=1}^j E(C/C_i)$ (since the inclusions $C \subseteq \oplus_{i=1}^j C/C_i \subseteq \oplus_{i=1}^j E(C/C_i)$ are essential) $= E(F)^j$ (since Ass$_R(C/C_i) = \{M\}$ for each i [4]). The result then follows since the decomposition of an injective module into a direct sum of modules of the form $E(R/P)$, $P \in$ Spec(R), is unique [4]. ∎

THEOREM 2.12. *Let B be a nonzero M-primary submodule of a finitely generated module A over a local ring (R, M). Then the following are equivalent*:

(2.12.1) *B is basically full in A.*

(2.12.2) *$B = (MB) :_A M$.*

(2.12.3) *$B = D :_A M$ for some submodule D of A.*

(2.12.4) $\mu(B) = \mathrm{sdim}(A/(MB))$.

(2.12.5) *There are $\mu(B)$ submodules of A in some (equivalently, each) irredundant representation of MB as an intersection of irreducible submodules of A.*

(2.12.6) $E(A/(MB)) \cong E(R/M)^{\mu(B)}$, *where $E(C)$ denotes an injective envelope of the R-module C.*

Proof. Assume that (2.12.1) holds and let $C = (MB) :_A M$. Then $MB \subseteq MC = M((MB) :_A M) \subseteq MB$. Therefore $MC = MB$, so $(MC) \cap B = (MB) \cap B = MB$, so since B is basically full in A it follows from Theorem 2.4 that $C = B$. Therefore $(MB) :_A M = B$, so (2.12.1) $\Rightarrow$ (2.12.2).

(2.12.2) $\Leftrightarrow$ (2.12.3) by Remark 2.10.

(2.12.2) $\Rightarrow$ (2.12.1). Let C be a submodule of A that properly contains B. Then to show that B is basically full in A it suffices by Theorem 2.4 to show that $MB \subset (MC) \cap B$.

For this, since B is M-primary and $B \subset C$, let $B = C_0 \subset C_1 \subset \cdots \subset C_n = C$ be a maximal chain of (M-primary) submodules between B and C. So $C_i = R(C_{i-1}, x_i)$ for some element $x_i \in C_i$ with $Mx_i \subseteq C_{i-1}$ for $i = 1, \ldots, n$. Therefore, in particular, $C_1 = R(B, x_1)$ is a cover of B (see Definition 2.9.2). If it is shown that $MB \subset (MC_1) \cap B$, then it follows that $MB \subset (MC) \cap B$, so it remains to show that $MB \subset (MC_1) \cap B$.

For this, suppose that $(MC_1) \cap B = MB$. Then $(Mx_1) \cap B \subseteq MB$. However, $C_1 = R(B, x_1)$ is a cover of B, so $Mx_1 \subseteq B$ and so it follows that $Mx_1 = (Mx_1) \cap B \subseteq MB$. Therefore $x_1 \in (MB) :_A M = B$ (by hypothesis), and this contradicts the fact that $B \subset C_1$. Therefore $MB \subset (MC_1) \cap B$, so $MB \subset (MC) \cap B$ and so B is basically full in A; hence (2.12.2) $\Rightarrow$ (2.12.1).

(2.12.2) $\Leftrightarrow$ (2.12.4). We have the inclusion $B/(MB) \subseteq ((MB) :_A M)/(MB)) \cong (0) :_{A/(MB)} M$ of R/M-vector spaces, and thus $\mu(B) = l_R(B/(MB)) \leq l_R(((MB) :_A M)/(MB)) = \mathrm{sdim}(A/(MB))$ with equality if and only if $B = (MB) :_A M$.

The equivalence of (2.12.4)–(2.12.6) follows from Lemma 2.11. ∎

Following [16], if $C = A/D$ for a submodule D of A, the number $j = t = \mathrm{sdim}(C)$ in Lemma 2.11 is called *the index of reducibility* of D; we denote it by $N(D)$. With this notation, we have the following corollary of Lemma 2.11 and Theorem 2.12.

COROLLARY 2.13. *Let B be a nonzero M-primary submodule of a finitely generated module A over a local ring (R, M). Then $\mu(B) \leq N(MB) = \operatorname{sdim}(A/(MB))$ with equality if and only if B is basically full.*

Proof. The equality $N(MB) = \operatorname{sdim}(A/(MB))$ follows from Lemma 2.11 with $C = A/(MB)$, and the inequality $\mu(B) \leq \operatorname{sdim}(A/(MB))$ follows from the proof of (2.12.2) $\Leftrightarrow$ (2.12.5). The last statement follows from the equivalence of (2.12.1), (2.12.5), and (2.12.6). ∎

We believe the equivalence of (2.12.1) and (2.12.6) to be of interest, because of the extensive literature concerning irredundant intersections of irreducible ideals; for example [6, 7, 9, 15, 16, 22] are a few of the many references on this topic. Proposition 2.16 and Section 6 have some additional results concerning the relationship between open irreducible ideals and basically full ideals.

The above corollary has the following analogue for M-full ideals. It follows from [29, Corollary to Theorems 1 and 2], but we give a direct proof.

PROPOSITION 2.14 [29]. *Let I be an M-primary ideal of a local ring (R, M) and let $x \in M$. Let $\overline{R} = R/(MI)$ and $\bar{x} = x + (MI)$, the image of x in $\overline{R}$. Then $\mu(I) \leq l_{\overline{R}}(\overline{R}/\bar{x}\overline{R})$ with equality if and only if I is M-full with $(MI) :_R xR = I$.*

Proof. It was shown in [29, Corollary to Theorem 1] that if (R, M) is an Artin local ring, then $\mu(I) \leq l_R(R/xR)$ for each ideal I and each $x \in M$. From the inclusions $MI \subseteq I \subseteq (MI) :_R xR \subseteq R, MI \subseteq (xR + MI) \subseteq R$, and the isomorphism $R/((MI) :_R xR) \cong (xR + MI)/(MI)$, we get $l_R(((MI) :_R xR)/I) + l_R(I/(MI)) = l_R(((MI) :_R xR)/(MI)) = l_R(R/(MI)) - l_R(R/((MI) :_R xR)) = l_R(R/(MI)) - l_R((xR + MI)/MI) = l_R(R/(xR + (MI))) = l_{\overline{R}}(\overline{R}/\bar{x}\overline{R})$. Thus $\mu(I) \leq l_{\overline{R}}(\overline{R}/\bar{x}\overline{R})$ and $l_R(((MI) :_R xR)/I) = 0$ if and only if $l_R(I/(MI)) = l_{\overline{R}}(\overline{R}/\bar{x}\overline{R})$. ∎

By a theorem of D. Rees [29, Theorem 5], every integrally closed ideal in an integrally closed integral domain is M-full. The following corollary is a variation of Rees' result.

COROLLARY 2.15. *Let B be a nonzero M-primary submodule of a finitely generated module A over a local ring (R, M). Then for each positive integer n the submodule $(M^n B) :_A M^n$ is basically full in A. In particular, if* $\operatorname{ht}(M) \geq 1$, *then the integral closure I_a of each M-primary ideal I of R is basically full.*

Proof. For each positive integer n let $B_n = (M^n B) :_A M^n$ and $C_n = (M^n B) :_A M^{n-1}$, so $B_n = C_n :_A M$. Also, $(MB_n) :_A M = [M(C_n :_A M)] :_A M = C_n :_A M = B_n$, by Remark 2.10; hence B_n is basically full in A by Theorem 2.12.

For the last statement, if $\mathrm{ht}(M) \geq 1$, [19, (3.2)] shows that $I \subseteq (M^n I) :_R M^n \subseteq I_a$ for all positive integers n, so it follows that $I_a \subseteq (MI_a) :_R M \subseteq (I_a)_a = I_a$; hence I_a is basically full by Theorem 2.12. ∎

PROPOSITION 2.16. *Let I be an open ideal in a local ring (R, M). Then MI is the irredundant intersection of $\mu(I)$ irreducible ideals in the following cases*: (*a*) *I is integrally closed*. (*b*) *I is of the form $(M^k K) :_R M^k$ for some open ideal K in R and some positive integer k*. (*c*) *$I = M^n$ for a large integer n* (*assuming that M is a regular ideal*). (*d*) *I is of the form $M^n K + (0) :_R M^k$ for some open ideal K in R and for some large integers n and k* (*assuming that* altitude$(R) > 0$).

Proof. It follows from Corollary 2.13 that MI is the irredundant intersection of $\mu(I)$ irreducible ideals if and only if I is basically full. Therefore (a) and (b) follow from Corollary 2.15. (c) follows from Theorem 7.2, and (d) follows from Theorem 7.4. ∎

Theorem 2.17 gives another characterization of the basically full submodules of a finitely generated R-module A; this characterization is in terms of covers (see Definition 2.9.2).

THEOREM 2.17. *Let B be a nonzero M-primary submodule of a finitely generated module A over a local ring (R, M). Then B is basically full in A if and only if $\mu(C) \leq \mu(B)$ for all covers C of B in A.*

Proof. If B is basically full in A, and C covers B, then $C = R(B, x)$ with $Mx \subseteq B$, and $MB \subset (MC) \cap B = MC$. Thus from $MB \subset MC \subseteq B \subset C$ and $l(C/B) = 1$, we get $l(C/(MC)) \leq l(B/(MB))$. That is, $\mu(C) \leq \mu(B)$.

For the converse assume that $\mu(C) \leq \mu(B)$ for all covers C of B, so $MB \subset (MC) \cap B$ for all covers C of B. Let D be a submodule of A that properly contains B. Then there exists a cover C of B such that $C \subseteq D$, so $MB \subset (MC) \cap B \subseteq (MD) \cap B$. Therefore $MB \subset (MD) \cap B$, so B is basically full in A by Theorem 2.4. ∎

3. BASICALLY FULL IDEALS AND EXTENSION RINGS

The main result in this section shows that one basically full ideal in a local ring yields other basically full ideals in related local rings.

Our first result shows that a basically full ideal in a factor ring of R lifts back to a basically full ideal in R.

PROPOSITION 3.1. *Let $K \subseteq I$ be ideals in a local ring (R, M) such that I is M-primary. If I/K is basically full in R/K, then I is basically full in R.*

Proof. If I/K is basically full in R/K, then $I/K = ((M/K)(I/K)) :_{R/K} (M/K)$, by Theorem 2.12.1 $\Rightarrow$ 2.12.2, so $I = (MI + K) :_R M$. But $(MI + K) :_R M \supseteq (MI) :_R M \supseteq I$, so $I = (MI) :_R M$. Hence I is basically full in R by Theorem 2.12.2 $\Rightarrow$ 2.12.1. ∎

Remark 3.2. The basically full closure I^* of a nonzero M-primary ideal I is defined in Definition 4.4 below to be the ideal $I^* = (MI) :_R M$. With this definition and Proposition 3.1 in mind, it is readily checked that $I^*/K \subseteq (I/K)^*$ for all ideals $K \subset I$ in R.

For a local ring (R, M), let X denote an indeterminate over R, and let $L := R[X]_{(M,\, X)R[X]}$.

THEOREM 3.3. *Let I be an open ideal in a local ring (R, M). Then:*

(3.3.1) *If J is an ideal in R such that $J \subseteq MI \subseteq I$, then I is basically full in R if and only if I/J is basically full in R/J.*

(3.3.2) *If (R', M') is a faithfully flat local ring extension of R such that $M' = MR'$, then I is basically full in R if and only if IR' is basically full in R'.*

(3.3.3) *I is basically full in R if and only if $(I, X)L$ is basically full in L.*

Proof. For (3.3.1), since $J \subseteq MI \subseteq I$, it follows that $(MI) :_R M = I$ if and only if $\overline{MI :_R M} = \overline{I}$, where the overbar denotes residue class modulo J. And [32, p. 148, (15)] shows that $\overline{MI :_R M} = (\overline{MI}) :_{\overline{R}} \overline{M}$. Also, $\overline{M}$ is the maximal ideal in $\overline{R}$, so it follows from Theorem 2.12.1 $\Leftrightarrow$ 2.12.2 that I is basically full in R if and only if $I/J = \overline{I}$ is basically full in $\overline{R} = R/J$.

For (3.3.2), $((MI) :_R M)R' = (MR'IR') :_{R'} MR'$, and MR' is the maximal ideal of R', so it follows from Theorem 2.12.1 $\Leftrightarrow$ 2.12.2 that I is basically full in R if and only if IR' is basically full in R'.

For (3.3.3), let $K = XL$ and $J = (I, X)L$. Then $K \subseteq J$, $L/K = R$, and $J/K = I$. Thus if I is basically full in R, then by Proposition 3.1, $(I, X)L$ is basically full in L.

Conversely, assume $(I, X)L$ is basically full in L. Let C be a cover of I. Then $(C, X)L$ is a cover of $(I, X)L$. So $\mu(C) + 1 = \mu(C, X)L \leq \mu(I, X)L = \mu(I) + 1$, where the inequality is by Theorem 2.17. Thus $\mu(C) \leq \mu(I)$, and the result follows by Theorem 2.17. ∎

4. THE BASICALLY FULL CLOSURE

Let (R, M) be a local ring and let A be a Noetherian R-module. In this section we show $B \mapsto B^* = (MB) :_A M$ is a semiprime operation on the set of nonzero M-primary submodules of A such that B^* is the smallest basically full submodule of A containing B.

We begin with the definitions.

DEFINITION 4.1. Let $\mathbf{S}$ be a set of submodules of an R-module A and let $B \mapsto B_\star$ be a mapping from $\mathbf{S}$ into $\mathbf{S}$. Consider the following conditions, where $B, C \in \mathbf{S}$ and I is an M-primary ideal of R: (a) $B \subseteq B_\star$; (b) if $B \subseteq C$, then $B_\star \subseteq C_\star$; (c) $(B_\star)_\star = B_\star$; (d) $IC_\star \subseteq (IC)_\star$. The mapping $B \mapsto B_\star$ is a *closure operation* on $\mathbf{S}$ if (a)–(c) hold for all submodules B, C in $\mathbf{S}$, and it is a *semiprime operation* if (a)–(d) hold for all submodules B, C in $\mathbf{S}$ and all M-primary ideals I of R.

In the case that $A = R$, the property (d) above is usually replaced by the property that $B_\star C_\star \subseteq (BC)_\star$. However, assuming conditions (a)–(c), it is easily seen that the formally weaker property (d) above is equivalent to the property that $B_\star C_\star \subseteq (BC)_\star$ for all M-primary ideals B and C of R. We have used the formally weaker property (d) because it readily carries over from ideals to modules.

THEOREM 4.2. *The mapping $B \mapsto B^* = (MB) :_A M$ on the set $\mathbf{S}$ of nonzero M-primary submodules of a Noetherian module A over a local ring (R, M) is a semiprime operation on $\mathbf{S}$ which takes each member B of $\mathbf{S}$ to the smallest basically full submodule of A that contains B.*

Proof. It is readily checked that (a) and (b) hold, and Remark 2.10 shows that (c) also holds.

For condition (d) let I be an M-primary ideal in R and $C \in \mathbf{S}$. Then $IC^* \subseteq (IC^*)^*$ (by (a)) $= [MI(MC :_R M)] :_R M \subseteq (MIC) :_R M = (IC)^*$.

The fact that B^* is the smallest basically full submodule of A that contains B follows from this and Theorem 2.12. ∎

COROLLARY 4.3. *If (R, M) is an Artinian local ring, then $(0) :_R M$ is the smallest basically full ideal in R.*

In view of Definition 4.1 and Theorem 4.2, we make the following definition.

DEFINITION 4.4. The *basically full closure* B^* of a nonzero M-primary submodule B of a Noetherian module A over a local ring (R, M) is the submodule $B^* = (MB) :_A M$.

Remark 4.5. It is noted in [18, Introduction] that every semiprime operation $I \mapsto I_\star$ on a set of ideals I of R satisfies $(I_\star J_\star)_\star = (IJ)_\star$; $(\sum_{i\in\Omega}(I_i)_\star)_\star = (\sum_{i\in\Omega} I_i)_\star$, and $(\cap_{i\in\Omega}(I_i)_\star)_\star = \cap_{i\in\Omega}(I_i)_\star$, where Ω is an arbitrary index set. Therefore it follows that the following hold for all M-primary ideals in a local ring (R, M): (a) $(I^*J^*)^* = (IJ)^*$; (b) $(\sum_{i\in\Omega}(I_i)^*)^* = (\sum_{i\in\Omega} I_i)^*$; and (c) $(\cap_{i\in\Omega}(I_i)^*)^* = \cap_{i\in\Omega}(I_i)^*$.

PROPOSITION 4.6. *Let B be an M-primary submodule of a Noetherian module A over a local ring (R, M). Then $(0_A) :_A M \subseteq B^*$, so $B^* = (B + ((0_A) :_A M))^*$ for every submodule B of A, where (0_A) is the zero submodule of A.*

Proof. Let $Z = (0_A) :_A M$. Then $MZ = (0_A)$, so $(B+Z)^* = [(M(B+Z)) :_A M] = (MB) :_A M = B^*$. ∎

Remark 4.7. Let (R, M) be a local ring. For a nonzero M-primary ideal I, the basically full closure $I^* = (MI) :_R M$ of I is obviously contained in $I :_R M$. This is helpful in the computation of examples.

5. BASICALLY FULL REDUCTIONS OF SUBMODULES

Recall that, for ideals $I \subseteq J$, I is said to be a *reduction* of J if $IJ^n = J^{n+1}$ for some positive integer n. With this definition in mind, in this section we introduce the concept of a basically full reduction of a nonzero M-primary submodule of a Noetherian R-module A. We show that most of the standard properties of ordinary reductions of ideals also hold for basically full reductions. (These results are similar to some of those in [20, 21] on delta-reductions and Ratliff–Rush reductions, respectively.) The main result, Theorem 5.6, characterizes the basically full closure B^* as the largest submodule C of A such that whenever D, E are R-submodules of A with $B \subseteq D \subseteq E \subseteq C$, then each minimal basis of D extends to a minimal basis of E.

For ideals $I \subseteq J$ in a Noetherian ring, it is well known that I is a reduction of J if and only if $J \subseteq I_a$, the integral closure of I. This motivates the following definition.

DEFINITION 5.1. Let (R, M) be a local ring and $C \subseteq D$ be nonzero M-primary submodules of a finitely generated R-module A.

(5.1.1) Then C is a *bf-reduction* of D if $C \subseteq D \subseteq C^*$. In this case we write $C \leq_{bf} D$. The module C is a *minimal bf-reduction* of D if it is minimal in the set of bf-reductions of D.

(5.1.2) C is *bf-basic* in case it has no proper bf-reductions.

A number of the elementary properties of bf-reductions will now be proved.

THEOREM 5.2. *Let (R, M) be a local ring, let A be a finitely generated R-module, and let C, D, E, C_i, and D_i be nonzero M-primary submodules of A.*

(5.2.1) *If $C \leq_{bf} D$, then $C^* = D^*$.*

(5.2.2) *If $C \leq_{bf} D$ and $D \leq_{bf} E$, then $C \leq_{bf} E$.*

(5.2.3) *If $C \leq_{bf} D$ and $C \subseteq E \subseteq D$, then $E \leq_{bf} D$ and $C \leq_{bf} E$. In particular, if I is an ideal in R, then $(C + ID) \leq_{bf} D$ and $C \leq_{bf} (C + ID)$.*

(5.2.4) *If $C \leq_{bf} D$, then the bf-reductions of D contained in C are the bf-reductions of C.*

(5.2.5) *If $C_1 \leq_{bf} D_1$ and $C_2 \leq_{bf} D_2$, then $(C_1 + C_2) \leq_{bf} (D_1 + D_2)$.*

(5.2.6) *If $C \leq_{bf} D$, then $IC \leq_{bf} ID$ for all M-primary ideals I in R such that IC is nonzero.*

Proof. For (5.2.1), if $C \leq_{bf} D$ then $C \subseteq D \subseteq C^*$, so $D^* = C^*$ by Definition 4.1 (b) and (c) (and Theorem 4.2).

For (5.2.2), the hypothesis and (5.2.1) imply that $C \subseteq D \subseteq C^* = D^*$ and $D \subseteq E \subseteq D^* = C^*$. Therefore $C \subseteq E \subseteq C^*$, so C is a bf-reduction of E by (5.1.1).

For (5.2.3), the hypothesis and (5.2.1) imply that $C \subseteq E \subseteq D \subseteq C^* = D^* = E^*$, so (5.1.1) shows that $E \leq_{bf} D$ and that $C \leq_{bf} E$. The last statement is now clear, since $C \subseteq C + ID \subseteq D$ for all ideals I of R.

For (5.2.4), let $H \subseteq C$ and assume that $C \leq_{bf} D$. If $H \leq_{bf} D$, then $H \leq_{bf} C$, by (5.2.3). And if $H \leq_{bf} C$, then $H \leq_{bf} D$, by (5.2.2).

For (5.2.5), the hypothesis implies that $C_1 + C_2 \subseteq D_1 + D_2 \subseteq (C_1)^* + (C_2)^*$, and $(C_1)^* + (C_2)^* \subseteq (C_1 + C_2)^*$ by Remark 4.5(b). So the conclusion follows from Definition 5.1.1.

For (5.2.6), if $C \subseteq D \subseteq C^*$, then $IC \subseteq ID \subseteq IC^*$, and $IC^* \subseteq (IC^*)^* = (IC)^*$ by Definition 4.1(d), so the conclusion follows from Definition 5.1.1. ∎

The next result characterizes bf-basic R-submodules of A (see Definition 5.1.2) as minimal bf-reductions.

PROPOSITION 5.3. *Assume that (R, M) is a local ring, A is a finitely generated R-module, and C is a nonzero M-primary submodule of A. Then C is bf-basic if and only if C is a minimal bf-reduction of some R-submodule D of A. If C is bf-basic and is a bf-reduction of D, then C is a minimal bf-reduction of D. So C is a minimal bf-reduction of every R-submodule between C and C^*.*

Proof. Since $C \leq_{bf} C$ by Definition 5.1.1, if C is bf-basic, then C is a minimal bf-reduction of C.

For the converse, assume that C is a minimal bf-reduction of D and let $E \leq_{bf} C$. Then $E \leq_{bf} D$ by Theorem 5.2.2, so $E = C$, by hypothesis; hence C is bf-basic by Definition 5.1.2.

Finally, the last two statements follow from what has already been shown (together with Definition 5.1.1). ∎

We next show that if R is a local (Noetherian) ring and A is a finitely generated R-module, then each nonzero M-primary R-submodule D of A has a minimal bf-reduction. To prove this we need (5.4.1) $\Leftrightarrow$ (5.4.4) of the following lemma. (Note that it is not assumed that C is M-primary in Lemma 5.4.4, but $(C + MD) \leq_{bf} D$ does imply that C is M-primary.)

LEMMA 5.4. *Let (R, M) be a local ring, let A be a finitely generated R-module, and let $C \subseteq D$ be nonzero submodules of A with D M-primary. The following statements are equivalent*:

(5.4.1) *C is M-primary and $C \leq_{bf} D$.*

(5.4.2) *C is M-primary and $C^* = D^*$.*

(5.4.3) *$MC = MD$.*

(5.4.4) *$(C + MD) \leq_{bf} D$.*

Proof. For (5.4.1) $\Leftrightarrow$ (5.4.2), if $C \leq_{bf} D$, then $C \subseteq D \subseteq C^*$, so $D^* = C^*$ by Definition 4.1(b) and (c). Conversely if $D^* = C^*$, then clearly $C \subseteq D \subseteq D^* = C^*$, so $C \leq_{bf} D$ by Definition 5.1.1.

For (5.4.2) $\Leftrightarrow$ (5.4.3), if $D^* = C^*$, then $MD \subseteq MD^* = MC^* = M((MC) :_A M) \subseteq MC \subseteq MD$, so $MC = MD$. Conversely, if $MC = MD$, then C is M-primary (since MD and M are) and $C^* = (MC) :_A M = (MD) :_A M = D^*$.

The implication (5.4.1) $\Rightarrow$ (5.4.4) follows from Theorem 5.2.3.

For (5.4.4) $\Rightarrow$ (5.4.3), if $(C + MD) \leq_{bf} D$, then $C + MD \subseteq D \subseteq (C + MD)^*$. Now $(C + MD)^* = M(C + MD) :_A M$, and the hypothesis implies that $C \subseteq D$, so $C \subseteq D \subseteq (C + MD)^*$ implies that $MC \subseteq MD \subseteq M[M(C + MD) :_A M] \subseteq MC + M^2 D$. Thus it follows from the lemma of Krull–Azumaya [14, (4.1)] that $MC = MD$. ∎

The proof of Theorem 5.5 is essentially the same as the proof of the analogous result for ordinary reductions [17, Theorem 1, p. 147] (but uses (5.4.1) $\Leftrightarrow$ (5.4.4) in place of [17, Lemma 2]), but we include it to show why it was necessary to not assume that C is M-primary in Lemma 5.4.4.

THEOREM 5.5. *Assume that (R, M) is a local ring, that A is a finitely generated R-module, and that C and D are nonzero M-primary submodules of A such that $C \leq_{bf} D$. Then there exists a minimal bf-reduction B of D such that $B \subseteq C$.*

Proof. Let Σ be the set of submodules of the form $G' + MD$, where $G' \subseteq C$ and $G' \leq_{bf} D$ (so G' is M-primary by the definition of a bf-reduction of D). Since $C + MD \in \Sigma$, Σ is not empty. Since the modules $(G' + MD)/MD$ are subspaces of the R/M vector space D/MD we may choose a bf-reduction G' of D such that $G' + MD$ is minimal in Σ. Let $x_1, \ldots, x_n \in G'$ be such that their images in $(G' + MD)/MD$ form a basis of $(G' + MD)/MD$ over R/M, and let $G = (x_1, \ldots, x_n)R$. Then $G \subseteq G' \subseteq C$ and $G + MD = G' + MD$. It follows by (5.4.1) $\Rightarrow$ (5.4.4), that $G + MD$ $(= G' + MD)$ is a bf-reduction of D. Thus, by (5.4.4) $\Rightarrow$ (5.4.1), G is M-primary and $G \leq_{bf} D$. Also, it follows that $MD \cap G = MG$.

Now if $G_0 \subseteq G$ is a bf-reduction of D, then $G_0 + MD \in \Sigma$ and $G_0 + MD \subseteq G + MD$. Therefore by the choice of G we have $G_0 + MD = G + MD = G' + MD$, so if $x \in G$, then $x = c_0 + z$ for some $c_0 \in G_0$ and $z \in MD$. Then $z = x - c_0 \in MD \cap G = MG$ (as noted at the end of the preceding paragraph), so $x \in G_0 + MG$. Therefore $G \subseteq G_0 + MG$, so $G \subseteq G_0$ by Nakayama's lemma. It therefore follows that $G_0 = G$, so G is a minimal bf-reduction of C. ∎

If $I \subseteq J \subseteq K$ are ideals of a local ring (R, M) and I is a minimal reduction of K, then by [17, Lemma 3, p. 147], each minimal basis of I extends to a minimal basis of J. However, it is not true in general that a minimal basis for J extends to a minimal basis for K. For example, if (R, M) is a regular local ring of altitude two and $M = (b, c)R$, then $I = (b^4, c^4)R \subseteq J = (b^4, b^3c^2, b^2c^3, c^4)R \subseteq K = (b^4, b^2c^2, c^4)R$ exhibit this behavior. The next result, which is a reformulation of [20, Theorem 5.2], shows that the stronger property that also every minimal basis for J extends to a minimal basis for K characterizes the basically full closure.

THEOREM 5.6. *Assume that (R, M) is a local ring, that A is a finitely generated R-module, and that $B \subseteq C$ are nonzero M-primary submodules of A. Then $B \leq_{bf} C$ if and only if whenever D, E are R-submodules of A with $B \subseteq D \subseteq E \subseteq C$, then each minimal basis of D extends to a minimal basis of E. Thus B^* is the unique largest submodule C of A such that whenever D, E are R-submodules of A with $B \subseteq D \subseteq E \subseteq C$, then each minimal basis of D extends to a minimal basis of E.*

Proof. Assume that $B \leq_{bf} C$, so that $B \subseteq C \subseteq B^* = (MB) :_A M$. Then $MB \subseteq MC \subseteq M(MB :_A M) \subseteq MB$, so $MB = MC$. Thus if D and E are R-submodules of A such that $B \subseteq D \subseteq E \subseteq C$, then $MD = ME$. It thus follows from Theorem 2.3 that every minimal basis of D can be extended to a minimal basis of E.

Conversely, assume that whenever D, E are submodules of A with $B \subseteq D \subseteq E \subseteq C$, then each minimal basis of D extends to a minimal basis of E. Let $b \in B - MB$. Then b can be extended to a minimal basis of B, and by hypothesis this minimal basis can be extended to a minimal basis of C; hence $b \notin MC \cap B$. Therefore $MC \cap B \subseteq MB$, and the opposite inclusion is clear, so $MC \cap B = MB$. Also, if $c \in C - B$, then let $D = B + Rc$ and let $b_1, \dots, b_g$ be a minimal basis of B. Then the hypothesis implies that $b_1, \dots, b_g$ can be (properly) extended to a minimal basis of D, and D is generated by $b_1, \dots, b_g, c$, so it follows that $b_1, \dots, b_g, c$ is a minimal basis of D. Therefore the hypothesis implies that these elements can be extended to a minimal basis of C, so it follows that $c \notin MC$. Thus, since

c is an arbitrary element in $C - B$, it follows that $MC \subseteq B$. However, it has already been shown that $MC \cap B = MB$. Thus $MC = MB$, and hence $C \subseteq (MB) :_A M = B^*$. ∎

Remark 5.7. Observe that Theorem 5.6 characterizes B^* as the *largest* submodule C of A that contains B and has the property that every minimal basis of each R-submodule D of A between B and C extends to a minimal basis of every R-submodule E of A between D and C. On the other hand, Theorem 4.2 characterizes B^* as the *smallest* submodule C of A that contains B and has the property that no minimal basis of C extends to a minimal basis of any R-submodule E of A that properly contains C.

6. BASICALLY FULL AND IRREDUCIBLE IDEALS

In this section we consider implications of the fact that the basically full closure B^* of an open irreducible submodule B is either B or the unique cover $B :_A M$ of B.

Throughout this section we use the fact that an irreducible submodule B has a unique cover, namely, $B :_A M$.

PROPOSITION 6.1. *Let A be a Noetherian module over a local ring (R, M) and let B be an open irreducible submodule of A. Then either $B^* = B$ or $B^* = B :_A M$ (= the unique cover of B).*

Proof. If $B \neq B^*$, then since B is an M-primary submodule of A, B^* contains a cover C of B, and since B is irreducible, $C = B :_A M$. Thus $B :_A M \subseteq B^* = (MB) :_A M \subseteq B :_A M$. ∎

In the remainder of this section we use $\operatorname{irr}(I)$ to denote the set of all irreducible ideals that appear in some irredundant decomposition of an ideal I into an intersection of irreducible ideals. Recall that $N(I)$ denotes the number of irreducible ideals that appear in any such decomposition of I. (See Lemma 2.11 and the paragraph preceding Corollary 2.13.)

COROLLARY 6.2. *Let $I \neq I^*$ be an open ideal in a local ring (R, M). Then for each ideal $Q \in \operatorname{irr}(MI)$ such that $I \subseteq Q$ it holds that the unique cover of Q is Q^*.*

Proof. It is shown in [9, (2.3)] that $Q \in \operatorname{irr}(MI)$ if and only if Q is maximal with respect to: (a) $MI \subseteq Q$; and, (b) I^* $(= (MI) :_R M) \not\subseteq Q$. Therefore fix $Q \in \operatorname{irr}(MI)$ and let $x \in I^* - Q$. Then $x \in Q^*$, since $I \subseteq Q$ implies that $I^* \subseteq Q^*$ (by Theorem 4.2), and $x \notin Q$. Therefore $Q \neq Q^*$; hence Q^* is the unique cover of Q by Proposition 6.1. ∎

PROPOSITION 6.3. *Let I be an open ideal in a local ring (R, M). Then the following hold*:

(6.3.1) *If $I \neq I^*$, then I is the irredundant intersection of $N(I)$ irreducible ideals Q such that Q^* is the unique cover of Q.*

(6.3.2) *If $I = I^*$ but $C \neq C^*$ for some cover C of I, then I is the irredundant intersection of $N(I)$ irreducible ideals Q such that, for $N(I) - 1$ of these Q, it holds that Q^* is the unique cover of Q.*

Proof. It is shown in [9, (3.2)] that I is the irredundant intersection of $N(I)$ ideals in $\operatorname{irr}(MI)$. Therefore (6.3.1) follows from Corollary 6.2.

For (6.3.2) let C be a cover of I such that $C \neq C^*$. Also, let $Q \in \operatorname{irr}(I)$ such that $C \not\subseteq Q$, so $I = Q \cap C$. By (6.3.1) let $C = Q_2 \cap \cdots \cap Q_k$, where each Q_i is irredundant and Q_i^* is the unique cover of Q_i. Then $Q \cap Q_2 \cap \cdots \cap Q_k$ is a (possibly redundant) decomposition of I into irreducible ideals, so the conclusion follows by omitting any redundant factors in this decomposition. ∎

It follows from Remark 4.5(c) that if $I = Q_1 \cap \cdots \cap Q_k$ and if $Q_i^* = Q_i$ for $i = 1, \ldots, k$, then $I = I^*$. On the other hand it can happen that $I = Q_1 \cap \cdots \cap Q_k$, where each Q_i is irreducible with Q_i properly contained in Q_i^* and yet $I = I^*$. We illustrate this in Example 6.4. We also show in Example 6.4 that if I is basically full, I may be an intersection of $N(I)$ irreducible ideals Q such that $Q \neq Q^*$ for $N(I) - 2$ of these ideals.

EXAMPLE 6.4. Let (R, M) be a regular local ring of altitude two, let $M = (b, c)R$, and let $n > 2$ be a positive integer. Then M^n is basically full by Remark 2.2.2 and each cover of M^n generated by monomials is basically full by Example 9.3, but $M^n = (b^n, c)R \cap (b^{n-2}, c^2)R \cap \cdots \cap (b, c^n)R$ is an irredundant decomposition of M^n into irreducible (in fact, parameter) ideals, and only the factors $(b^n, c)R$ and $(b, c^n)R$ are basically full, by Remark 9.4. Moreover, this is the only factorization of M^n into monomial parameter ideals. However, the representation

$$M^3 = (b^2, c^2)R \cap (b^2 + bc, c^2)R \cap (b^2, bc + c^2)R$$

is an irredundant representation of M^3 as an intersection of irreducible ideals Q_i, where Q_i is properly contained in Q_i^* for each i.

Proof. It is shown in [8, Theorem 2.4] that M^n is the irredundant intersection of the $N(I) = n$ ideals $(b^i, c^{n+1-i})R$ for $i = 1, \ldots, n$ (and these ideals are irreducible, since b, c is an R-sequence). Then [8, Theorem 4.10] shows that this is the only such monomial parameter decomposition of M^n. ∎

7. WHEN IS M^n BASICALLY FULL?

It is mentioned in Remark 2.2.2 that every power M^n of the maximal ideal M in a regular local ring of altitude two is basically full. In this section we first consider conditions in order that all powers (or, all large powers) of M are basically full in an arbitrary local ring. That is, we restrict attention to the case where our module A is the ring R and the submodule B is a power M^n of the maximal ideal M of R. We determine necessary and sufficient conditions for M^n to be basically full for all (resp., for all large) positive integers n. We then show that $M^nI + ((0) :_R M^k)$ is basically full for all open ideals I and for all large integers n and k, and then close this section by showing that every nonzero open ideal in R is basically full if and only if R is a principal ideal ring.

Recall that by Theorem 4.2, M^n is basically full if and only if $M^n = M^{n*}$, where $M^{n*} = M^{n+1} :_R M$. We use this to characterize when M^n is basically full for every positive integer n. (One of these conditions is that M^n is M-full for every positive integer n. However, in general it is not true that every basically full ideal is M-full, as is shown in Example 9.1.)

THEOREM 7.1. *Let (R, M) be a local ring. The following conditions are equivalent*:

(7.1.1) *M^n is basically full for every positive integer n.*

(7.1.2) *M^n is M-full for every positive integer n.*

(7.1.3) Grade$(G^+(M)) > 0$, *where $G(M) = R/M \oplus M/M^2 \oplus \cdots$ and $G^+(M)$ is its maximal homogeneous ideal.*

Proof. Let X be an indeterminate and let $R(X) := R[X]_{MR[X]}$. Then M^n is M-full if and only if $M^nR(X)$ is M-full (by the definition in Remark 2.2.1), M^n is basically full if and only if $M^nR(X)$ is basically full (by Theorem 3.3.2), and grade$(G^+(M)) > 0$ if and only if grade$(G^+(MR(X))) > 0$, so it may be assumed that R/M is infinite.

It is shown in [23, Theorem 2.1] that $G(M) = \mathbf{R}/U$, where $\mathbf{R} = R[u, tM]$ and $U = u\mathbf{R}$. (Here, t is an indeterminate and $u = 1/t$, so $\mathbf{R}$ is the Rees ring of R with respect to M.) Therefore grade$(G^+(M)) > 0$ if and only if there is an $\mathbf{R}$-sequence of the form u, t^kb (and since R/M is infinite it may be assumed that $k = 1$) in $\mathbf{R}$. And it is shown in [27, Theorem 2.5] that u, tb is an $\mathbf{R}$-sequence if and only if $M^{n+1} :_R bR = M^n$ for all positive integers n. However, $M^{n+1} :_R bR = (MM^n) :_R bR$, so it follows from the definition of M-full (see Remark 2.2.1) that (7.1.3) $\Leftrightarrow$ (7.1.2).

It is noted in Remark 2.2.1 that (7.1.2) $\Rightarrow$ (7.1.1).

Finally, assume that (7.1.1) holds, so

$$(*)\quad M^{i+2} :_R M = M^{i+1} \text{ for all nonnegative integers } i,$$

by Theorem 4.2. Also, with $\mathbf{R} = R[u, tM]$, it is readily checked that $u\mathbf{R} = \sum_i M^{i+1}t^i$ and that $u\mathbf{R} :_{\mathbf{R}} (tM\mathbf{R}) = u^2\mathbf{R} :_{\mathbf{R}} (M\mathbf{R}) = \sum_i (M^{i+2} :_R M) \cap M^i)t^i$ (with the convention that $M^i = R$ if $i \leq 0$). Therefore it follows from $(*)$ that $u\mathbf{R} = u\mathbf{R} :_{\mathbf{R}} (tM\mathbf{R})$, so since $G(M) = \mathbf{R}/u\mathbf{R}$, it follows that $\text{grade}(G^+(M)) > 0$; hence (7.1.1) $\Rightarrow$ (7.1.3). ∎

It can happen that $M^{n+1} :_R M \supset M^n$ in an altitude one local domain. For example, let t be an indeterminate over a field k. Then $R = k[t^4, t^5, t^{11}]$ localized at (t^4, t^5, t^{11}) has the property that $M^3 :_R M \supset M^2$. So M^2 is not basically full, although M is clearly basically full.

In the proof of Theorem 7.2, we use the concept of a superficial element, where an element x in R is a *superficial element of order* s for an ideal I in case there exists a positive integer c such that $(I^n :_R xR) \cap I^c = I^{n-s}$ for all large positive integers n [33, p. 285]. If R/M is infinite, then I has a superficial element x of order 1, and if $\text{grade}(I) > 0$, then x may be chosen to be a regular element [14, (22.1) and (22.2)]. It follows from [14, (3.12)] that $I^n :_R xR \subseteq ((0) :_R xR) + I^c$ for all large integers n. Thus if $\text{grade}(I) > 0$, then $I^n :_R xR = I^{n-1}$ for all large integers n. With this definition and fact in mind, we determine necessary and sufficient conditions for all large powers of M to be basically full. We also show this is equivalent to all large powers of M being M-full.

THEOREM 7.2. *Let (R, M) be a local ring. The following conditions are equivalent*:

(7.2.1) *M^n is basically full for all large integers n.*

(7.2.2) *M^n is M-full for all large integers n.*

(7.2.3) Grade$(M) > 0$.

Proof. As in the first paragraph of the proof of Theorem 7.1 it may be assumed that R/M is infinite.

Assume that (7.2.3) holds. Then M is a regular ideal, so it follows from [14, (3.12)] (mentioned preceding this theorem) that M has a superficial element x of order 1 such that $M^n :_R xR = M^{n-1}$ for all large integers n. Therefore $(MM^{n-1}) :_R xR = M^{n-1}$, so it follows from the definition of M-full (see Remark 2.2.1) that M^n is M-full for all large integers n. Therefore (7.2.3) $\Rightarrow$ (7.2.2).

Also, Remark 2.2.1 shows that (7.2.2) $\Rightarrow$ (7.2.1).

To complete the proof it must be shown that (7.2.1) $\Rightarrow$ (7.2.3). For this, it is shown in Proposition 4.6 that $Z \subseteq M^{n*}$ for all positive integers n, where $Z = (0) :_R M$ and where $M^{n*} = (MM^n) :_R M$. Therefore, if (7.2.3) does not hold, then $Z \neq (0)$, so let k be a positive integer such that $Z \not\subseteq M^k$. Then $Z \subseteq M^{n*}$ and $Z \not\subseteq M^n$ for all integers $n \geq k$, so it follows that M^n is not basically full for all integers $n \geq k$; hence it follows that (7.2.1) $\Rightarrow$ (7.2.3). ∎

COROLLARY 7.3. *Let (R, M) be a local ring of altitude $d > 0$. Then $M^n + ((0) :_R M^k)$ is basically full for all large integers n and k.*

Proof. Let k be large enough that $(0) :_R M^k = (0) :_R M^m$ for all integers $m \geq k$ and let $Z = (0) :_R M^k$. Then M/Z is a regular ideal in R/Z and altitude$(R/Z) > 0$, so it follows from (7.2.2) $\Leftrightarrow$ (7.2.3) that $M^n/Z = (M/Z)^n$ is basically full for all large integers n. Therefore it follows from Proposition 3.1 that $M^n + Z$ is basically full for all large integers n. ∎

Our next result extends Corollary 7.3 to $(M^nI) + ((0) :_R M^k)$ for an arbitrary open ideal I.

THEOREM 7.4. *Let (R, M) be a local ring of altitude $d > 0$ and let I be an open ideal in R. Then $(M^nI) + ((0) :_R M^k)$ is basically full for all large integers n and k.*

Proof. Assume that it is known that the conclusion holds in the case where grade$(M) > 0$. Then for the general case let $Z = (0) :_R M^k$ with k large, so grade$(M/Z) > 0$, so $\overline{M^nI}$ is basically full in R/Z for all large integers n, and so it follows by Proposition 3.1 that $(M^nI) + Z$ is basically full for all large integers n. Therefore it may be assumed that M is a regular ideal, and it remains to show that M^nI is basically full for all large integers n.

For this, M^nI is basically full in R if and only if $M^nIR(X)$ is basically full in $R(X) = R[X]_{MR[X]}$, by Theorem 3.3.2, so it may be assumed that R/M is infinite.

It is shown in [13] that if I is an arbitrary ideal in R, then there exist a (regular) element x in M and a positive integer c such that $((M^nI) :_R xR) \cap (M^cI) = M^{n-1}I$ for all large integers n. Also, it follows from [14, (3.12)] that $(M^nI) :_R xR \subseteq M^cI$ for all large integers n, so $(M^nI) :_R xR = M^{n-1}I$ for all large integers n. However, $(M^nI) \subseteq (MM^nI) :_R M \subseteq (MM^nI) :_R xR = M^nI$, so $M^nI = (MM^nI) :_R M$ for all large integers n; hence if I is M-primary, then M^nI is basically full by Theorem 2.12. ∎

We next show that every nonzero open ideal in R is basically full if and only if R is a principal ideal ring.

THEOREM 7.5. *Let (R, M) be a local ring. A necessary and sufficient condition for every nonzero M-primary ideal of R to be basically full is that M is principal and thus that R is a principal ideal ring.*

Proof. If M is principal, it is clear that all nonzero M-primary ideals are basically full.

Conversely, assume all nonzero M-primary ideals are basically full. If I and J are nonzero M-primary ideals and if $MI = MJ$, then $I = (MI) :_R M = (MJ) :_R M = J$. Thus $I \neq J$ implies $MI \neq MJ$. Consider first the case where altitude(R) > 0. For each positive integer n the ideal M^n is the only reduction of itself (since each ideal is basically full and since it is shown in [17, Lemma 3, p. 147] that a minimal basis of a minimal reduction of an ideal can be extended to a minimal basis of the ideal). Even if the field R/M is finite, there exists a positive integer k such that M^{nk} is generated by analytically independent elements for each n. It follows that altitude(R) = 1 and some power of M is principal. Thus M is invertible. Since R is local, M is principal and R is a DVR [14, (12.1)]. Therefore it may be assumed that altitude(R) = 0.

It may clearly be assumed that R is not a field. Then the hypothesis implies that the socle $(0) :_R M$ of R is the unique nonzero ideal I of R such that $MI = (0)$. Hence $S = (0) :_R M$ is principal. If $S \subset M$, then for $x \in (S :_R M) - S$, we have $Mx = S$. It follows that $xR = (S :_R M) = (0) :_R M^2$ is principal. If $xR \subset M$, then for $y \in (xR :_R M) - xR$, we have $My = xR$. Hence $yR = (xR :_R M) = (0) :_R M^3$. Continuing this process, we see that M is principal. ∎

Remark 7.6. Since every ideal in R is a finite intersection of irreducible ideals, it follows from Remark 4.5(c) and Theorem 7.5 that R is a principal ideal ring if and only if every nonzero irreducible M-primary ideal in R is basically full.

8. BASICALLY FULL MONOMIAL IDEALS

In this section we observe that if I is an M-primary ideal in a regular local ring (R, M), which is generated by monomials in a fixed regular system of parameters of R, then the basically full closure I^* of I is also a monomial ideal. Moreover, to check if I is basically full, it suffices to show that a minimal monomial basis of I does not extend to a minimal monomial basis of any properly larger monomial ideal of R.

Let $x_1, x_2, \ldots, x_n$ be an R-sequence in a local ring (R, M). The following is an immediate consequence of [28, Theorem 1].

LEMMA 8.1. *If* $f, f_1, f_2, \ldots, f_k$ *are monomials in* $x_1, \ldots, x_n$ *and* $f \in (f_1, f_2, \ldots, f_k)R$, *then* $f \in f_i A$ *for some* $i \in \{1, \ldots, k\}$.

Thus if we identify each nonzero monomial $x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ with the n-tuple of nonnegative integers $(a_1, a_2, \ldots, a_n)$, and partially order the set $M(e_1, \ldots, e_n)$ of these n-tuples by defining $(a_1, a_2, \ldots, a_n) \leq (b_1, b_2, \ldots, b_n)$ in $M(e_1, \ldots, e_n)$ if $a_i \leq b_i$ for each i, then each monomial ideal in $x_1, x_2, \ldots, x_n$ is generated by a unique antichain in the finite poset $M(e_1, \ldots, e_n)$, where by definition an *antichain* in $M(e_1, \ldots, e_n)$ is a set of pairwise incomparable elements in $M(e_1, \ldots, e_n)$.

PROPOSITION 8.2. *Let (R, M) be a regular local ring of altitude d, let $M = (x_1, \ldots, x_d)R$, and let I be an M-primary ideal that is generated by monomials in $x_1, \ldots, x_d$. Then the basically full closure I^* of I is also generated by monomials in $x_1, \ldots, x_d$.*

Proof. Since $I^* = (MI) :_R M$, this is immediate. ∎

PROPOSITION 8.3. *Let (R, M) be a regular local ring of altitude d, let $M = (x_1, \ldots, x_d)R$, and let I be an ideal of R that is generated by monomials in $x_1, \ldots, x_d$. The following are equivalent*:

(8.3.1) *I is basically full.*

(8.3.2) *No minimal basis for an ideal $J \supset I$ generated by monomials in $x_1, \ldots, x_d$ extends a minimal basis of I.*

(8.3.3) *$(MJ) \cap I \supset MI$ for each ideal $J \supset I$ generated by monomials in $x_1, \ldots, x_d$.*

Proof. It follows from Theorem 2.3 that (8.3.1) $\Rightarrow$ (8.3.2) $\Leftrightarrow$ (8.3.3).

To show that (8.3.3) $\Rightarrow$ (8.3.1), let $I^* = (MI) :_R M$. Then I^* is a monomial ideal and $MI \subseteq MI^* = M((MI) :_R M) \subseteq MI$. Therefore $MI^* = MI$, so $(MI^*) \cap I = (MI) \cap I = MI$; hence $I^* = I$ by (8.3.3). That is, $(MI) :_R M = I$, and therefore I is basically full by Theorem 2.12 (or by Section 4). ∎

COROLLARY 8.4. *Let (R, M) be a regular local ring of altitude d, let $M = (x_1, \ldots, x_d)R$, and let I be an ideal of R that is generated by monomials in $x_1, \ldots, x_d$. Then I is basically full if and only if $\mu(J) \leq \mu(I)$ for each cover J of I that is generated by monomials in $x_1, \ldots, x_d$.*

Proof. This follows immediately from (8.3.1) $\Leftrightarrow$ (8.3.3). ∎

PROPOSITION 8.5. *Let (R, M) be a regular local ring of altitude d, let $M = (x_1, \ldots, x_d)R$, and let I be an ideal of R that is generated by monomials in $x_1, \ldots, x_d$. Then I is basically full if and only if I is generated by a maximal antichain.*

Proof. ($\Rightarrow$) If the unique antichain A generating I is part of a strictly larger antichain L, then the elements of L are a minimal basis for a larger ideal J extending the given antichain. Thus I is not basically full.

($\Leftarrow$) If the unique antichain A generating I is maximal, then A is a minimal basis of I and is not part of a minimal basis of monomials generating an ideal J, which is strictly larger than I. Thus I is basically full by Proposition 8.3. ∎

9. SOME EXAMPLES

In this section we give some examples that illustrate our results. These examples are all in a regular local ring R of altitude two, and they all concern monomial ideals (in the generators b, c of the maximal ideal M of R), so we *fix the following notation for this section*: (R, M) is a regular local ring of altitude two and $M = (b, c)R$.

Our first example is of basically full ideals that are not M-full.

EXAMPLE 9.1. Let $n > 2$ be a positive integer. Let $I = (b^n, b^{n-1}c^{n-1}, c^n)R$. Then the exponents $(n, 0), (n-1, n-1), (0, n)$ are a maximal antichain in $\mathbb{N} \times \mathbb{N}$, and thus I is basically full by Proposition 8.5, but I clearly does not have the Rees property mentioned in Example 2.2. Thus I is not M-full.

The next two examples show that there are "many" basically full ideals in R. The first of these shows that the length of M^n can be computed using a saturated chain of ideals each of which is basically full.

EXAMPLE 9.2. Let n be a positive integer, let $I_0 = M^n$, and for $i = 1, \ldots, n$ let $I_i = (I_{i-1}, b^{n-i}c^{i-1})R$. Then $M^n = I_0 \subset I_1 \subset \cdots \subset I_n = M^{n-1}$, I_{i+1} is a cover of I_i for $i = 0, \ldots, n-1$, and each I_i is basically full. Therefore the length of M^n can be computed by a saturated chain of basically full ideals.

Proof. It was noted in Remark 2.2.2 that $I_0 = M^n$ and $I_n = M^{n-1}$ are basically full.

To see that I_{i+1} is a cover of I_i, note that $I_1 = (b^{n-1}, b^{n-2}c^2, b^{n-3}c^3, \ldots, c^n)R$, $I_2 = (b^{n-1}, b^{n-2}c, b^{n-3}c^3, \ldots, c^n)R, \ldots, I_i = (b^{n-1}, b^{n-2}c, \ldots, b^{n-i}, c^{i-1}, b^{n-i-1}c^{i+1}, \ldots, c^n)R, \ldots, I_{n-1} = (b^{n-1}, b^{n-2}c, \ldots, bc^{n-2}, c^n)R$, and $I_n = M^{n-1}$. Therefore it readily follows that I_{i+1} is a cover of I_i for $i = 0, \ldots, n-1$.

To see that each I_i is basically full, note that each I_i is generated by monomials (in b, c), so it follows that the displayed basis of I_i is a minimal basis, so $\mu(I_i) = n$. Now a minimal basis of I_i can be extended to its basically full closure $(MI_i) :_R M$, since $(M((MI_i) :_R M)) \cap I_i = (MI_i) \cap I_i = MI_i$, so $\mu((MI_i) :_R M) \geq \mu(I_i) = n$. However, $M^n \subset I_i \subseteq (MI_i) :_R M$, so $\mu((MI_i) :_R M) < \mu(M^n) = n + 1$, by [29, Theorem 4], so it follows that $I_i = (MI_i) :_R M$; hence I_i is basically full by Theorem 2.12.

For the last statement in the example, it follows from what has already been proved that there is a chain of basically full ideals from M to M^n of length equal to the length of M^n. ∎

In the proof of Example 9.2, each I_i is integrally closed by [2, p. 140, Exercise 4.23] (except this exercise refers to indeterminates instead of elements of a regular system of parameters), and thus is a basically full ideal by Corollary 2.15. In fact each I_i is M-full by a theorem of D. Rees [29, Theorem 5] (which is mentioned just before Theorem 2.15).

With the preceding example in mind, it should be noted that not all open basically full monomial ideals can have their lengths computed by using only basically full ideals. For example, the length of the ideals in Example 9.1 cannot be computed this way.

The next example shows that each monomial cover of M^n is basically full. Concerning this result the stronger result (2.15) that all covers of M^n are integrally closed is proved in [10, Example 2.2]. (In [10] the authors assume that R/M is infinite, but this assumption is not needed for this particular result.)

EXAMPLE 9.3. Let n be a positive integer. Then each cover of M^n that is generated by monomials (in b, c) is basically full.

Proof. Let J be a cover of M^n that is generated by monomials in b, c. Then J must be of the form $(M^n, b^{n-i}c^{i-1})R$ for some integer $i \in \{1, \dots, n\}$ (since $MJ \subseteq M^n$), so J must have a minimal basis of n monomials in b, c (since M^n has a minimal basis of $n+1$ monomials). Therefore each monomial cover K of J has a minimal basis of at most $n+1$ monomials, and since $M^n \subset J \subset K$ it follows that $\mu(K) < \mu(M^n) = n+1$, by [29, Theorem 4]. Therefore $\mu(K) \leq \mu(J)$, so J is basically full by Corollary 8.4. ∎

It follows from Theorem 7.4 that M^nI is basically full for large integers n and for all open ideals I in a regular local ring. The next example shows that I and M^nI may be basically full, but that MI need not be. For this example we use the following remark.

Remark 9.4. Let $x_1, \dots, x_g$ be monomials (in b, c) that are a minimal basis of an open ideal I, arranged in decreasing powers of b (so arranged in increasing powers of c). (So there exist positive integers m and n such that $x_1 = b^m$; $x_g = c^n$; and, $\min\{m, n\} \geq g-1$ (and $\min\{m, n\} = 1$ if and only if $g = 2$.) Then I is basically full if and only if the following holds for $h = 1, \dots, g-1$: if $x_h = b^ic^j$, then either $x_{h+1} = b^{i-1}c^{j+k}$ for some $k \in \{1, \dots, n-j\}$ (and $k = n-j$ if and only if $h = g-1$ and $i = 1$), or $x_{h+1} = b^{i-k}c^{j+1}$ for some $k \in \{1, \dots, i\}$ (and $k = i$ if and only if $h = g-1$ and $j = n-1$).

Proof. By the parenthetical sentence (concerning x_1 and x_g) it may be assumed that $g > 2$. Then the conclusion follows readily from the fact that a set of monomial generators of an open monomial ideal generate a basically full ideal if and only if they are a maximal antichain, by Proposition 8.5. ∎

EXAMPLE 9.5. Let $n > 2$ be a positive integer, and let $I = (b^n, b^{n-1}c^{n-1}, c^n)R$. Then I and M^kI are basically full for all large integers k, but MI is not basically full.

Proof. It is shown in Example 9.1 that I is basically full, and Theorem 7.4 shows that $M^k I$ is basically full for all large integers k. However, $MI = (b^{n+1}, b^n c, bc^n, c^{n+1})R$, so since $n \geq 3$ it follows from Remark 9.4 that MI is not basically full. ∎

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